Lecture 11

Master equation for the damped H.O.

[See e.g.: "Stochastic Methods in Quantum Optics" CHARMICHAEL]

One can also reconsider the problem of a H.O. interacting with a heat bath by using the density matrix ρ.

$$\rho_{SB} = \rho_S \otimes \rho_{BATH}$$

The master equation can be derived from the QSDE.

starting point $\hat{H} = \underbrace{\hat{H}_S + \hat{H}_B}_{H_0} + \hat{H}_{int}$

$\hat{H}_S = \hbar\omega \hat{a}^\dagger \hat{a}$

$\hat{H}_B = \sum_k \hbar\omega_k (\hat{b}_k^\dagger \hat{b}_k)$

$\hat{H}_{int} = \sum_k \hbar g_k \hat{a} \hat{b}_k^\dagger + h.c.$

Approximations:

(1) oscillators in bath tightly spaced $\sum_k \to \int d\omega D(\omega)$

(2) Large bath in a state of equilibrium. Perturbations of bath negligible

$\rho_{BATH}(t) = \rho_{BATH}(0)$ ie $\rho = \rho_S(t) \otimes \rho_{BATH}(0)$

(3) Counter-rotating terms are neglected

starting point is the Schrödinger equation.

$$i\hbar \frac{\partial \rho_{SB}}{dt} = [\hat{H}, \rho_{SB}]$$

Transform to a rotating frame (interaction picture) i.e separate dynamics of $H_R + H_S$ from H_{SR} dynamics

$$\tilde{\rho}_{SB} = \exp\left(\frac{iH_0}{\hbar} t\right) \rho_{SB} \exp\left(-\frac{iH_0}{\hbar} t\right)$$

Thus

$$i\hbar \frac{d}{dt} \tilde{\rho}_{SB} = [\tilde{\hat{H}}_{int}, \tilde{\rho}_{SB}]$$

$\tilde{H}_{int}$ ist the transformed $\hat{H}_{int}$.

$$\tilde{\hat{H}}_{int}(t) = \sum_k \hbar g \left(\hat{a}^\dagger \hat{b}_k e^{i(\omega-\omega_k)t} + \hat{a} \hat{b}_k^\dagger e^{-i(\omega-\omega_k)t}\right)$$

$\left(\hat{a} \to \hat{a} e^{+i\omega t} \text{ transformation} \right)$

Next solve

$$\tilde{\rho}_{SB}(t) = \tilde{\rho}_{SB}(0) + \frac{1}{i\hbar} \int_0^t dt' [\tilde{H}_{int}(t'), \rho(t')] .$$

substitute $\tilde{\rho}_{SB}(t)$ back into equation

$$\dot{\tilde{\rho}}_{SB}(t) = \frac{1}{i\hbar}[\tilde{H}_{SB}(t), \rho(t)] = \frac{1}{i\hbar}[\tilde{H}_{SB}(t), \rho(0)] + \frac{1}{(i\hbar)^2} \int_0^t dt' [\tilde{H}_{SB}(t), [\tilde{H}_{SB}(t'), \rho(t')]]$$

Next assume that reservoir is in a thermal state and trace over reservoir ie $\rho_S = Tr_B \rho_{SR} = \sum_b \langle b| \rho_{SB} |b\rangle$ and $Tr_B [\tilde{H}_{SB}(t) \rho_S(0)] = 0$

assume that $\rho_{SB} = \rho_S(t) \rho_B(0)$ i.e. $\rho_B(t) = \rho_R(0)$ (bath unaffected by system)

$$\dot{\tilde{\rho}}_S = \frac{1}{\hbar^2} \int_0^t dt' \, Tr_R \{ [\hat{H}_{SB}(t'), [H_{SB}(t'), \rho_{SB}(t')]] \}$$

$\mathrm{Tr}\,[\hat{H}_{SB}(t)\,\varrho_B(0)] = 0$ when coupling operators of reservoir have zero mean e.g. $\langle \hat{b}_k \rangle = \langle \hat{b}_k^+ \rangle = 0$

Next we apply the BORN APPROXIMATION: $(\tilde{\varrho}_{SB}(t) = \tilde{\varrho}_S \otimes \varrho_B(0) + \text{higher order})$

$$\boxed{\dot{\tilde{\varrho}} = -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_B[\hat{H}_{SB}(t'),[\tilde{H}_{SB}(t'),\tilde{\varrho}_S(t')\otimes\tilde{\varrho}_B(0)]]}$$

Next apply the Markov-approximation

$$\boxed{\dot{\varrho} = -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_B[\hat{H}_{SB}(t'),[\tilde{H}_{SB}(t'),\underline{\varrho_S(t)}\otimes\varrho_B(0)]]}$$

Thus $\varrho_S(t') \to \varrho_S(t)$

The system thus now becomes memoryless & does not depend on the past time $t' < t$

Lets consider specific example of coupling terms: $\hat{H}_{SB}(t) = \hbar\sum g_k \hat{a}\hat{b}_k^+ + h.c$

$\equiv \hbar(\hat{a}\hat{T}^+ + \hat{a}^+\hat{T})$

(Energy relaxation)

Hence transformed interaction:

$$e^{+i\omega_0\hat{a}^+\hat{a}t}\,\hat{a}\,e^{-i\omega_0\hat{a}^+\hat{a}t} = \hat{a}\,e^{-i\omega_0 t}$$

$$e^{\sum_k i\omega_k\hat{b}_k^+\hat{b}_k}\left(\sum_k \hbar g_k \hat{b}_k\right)e^{\sum_k i\omega_k\hat{b}_k^+\hat{b}_k} = \sum_k \hbar g_k \hat{b}_k e^{-i\omega_k t}$$

Thus:

$$\boxed{\tilde{H}_{SB}(t) = \sum_k \hbar g_k\left(\hat{a}^+\hat{b}_k e^{(\omega-\omega_k)it} + \hat{a}\hat{b}_k^+ e^{-(\omega-\omega_k)it}\right)}$$

interaction Hamiltonian rotating frame

inserting yields now 16 Terms (tedious calculation) due to double commutator

Rewriting the coupling as:

$$\hat{H}_{SB} = \sum_k \hbar g_k \underbrace{(\hat{a}^+\hat{b}_k)e^{i(\omega-\omega_k)t}}_{(1)} + h.c$$

$$= \hbar\sum_{i=1,2} \underbrace{\hat{S}_i(t)}_{\text{system}}\,\underbrace{\hat{T}_i(t)}_{\text{Bath}} \qquad \hat{T}_i(t) \equiv \sum_k \hat{b}_k e^{-i(\omega-\omega_k)t} g_k$$

(introduced for convenience)

The Master equation becomes:

$$\dot{\varrho} = -\sum_{i,j}\int_0^t \mathrm{tr}_B\{[\tilde{S}_i(t)\tilde{T}_i(t),[\tilde{S}_j(t')\tilde{T}_j(t'),\varrho_S(t)\otimes\varrho(0)]\,dt'$$

$$= -\sum_{i,j}\int_0^t dt'\,\{\tilde{S}_i(t)\tilde{S}_j(t')\varrho(t)\,\mathrm{Tr}_B\{T_i(t)\varrho_B(0)T_j(t')\}$$

$$- \tilde{S}_i(t)\tilde{\varrho}(t)\tilde{S}_j(t')\,\mathrm{Tr}\{T_i(t)\varrho_B(0)T_j(t')\} - S_j(t')\varrho(t)S_i(t)\times$$

$$\mathrm{Tr}\{\tilde{T}_j(t')\varrho_B(0)T_i(t)\} + \varrho(t)S_j(t')S_i(t)\,\mathrm{Tr}\{\varrho_B T_j(t')T_i(t)\}$$

$$\dot{\tilde{\rho}}_s = -\sum_{ij} \int_0^t dt' \left\{ \left(s_i(t) s_j(t') \tilde{\rho}_s(t) - s_j(t) \tilde{\rho}_s(t) s_i(t) \right) \langle T_i(t) T_j(t') \rangle_R \right.$$

$$\left. + \left(\tilde{\rho}_s(t) s_j(t') s(t) - s_i(t) \rho_s(t) s_j(t') \right) \langle T_j(t') T_i(t) \rangle_R \right.$$

and we have used $tr\{\hat{A}\hat{B}\hat{C}\} = tr\{\hat{C}\hat{A}\hat{B}\}$ (Cyclic property of trace)

where $\langle \tilde{T}_i(t) \tilde{T}_j(t') \rangle_R = Tr_B \{ \rho_B(0) \tilde{T}_i(t) \tilde{T}_j(t') \}$ correlations of the reservoir

remember that now $s_1(t) = \hat{a} e^{-i\omega_0 t}$ $s_2 = \hat{a}^+ e^{+i\omega_0 t}$ and $T_1 = \sum g_k \hat{b}_k e^{-i(\omega - \omega_k)t}$, $T_2 = \sum_k g_k \hat{b}_k^+ e^{+i(\omega - \omega_k)t}$

The next approximation is to assume that the reservoir dynamics is fast compared to the system evolution.

$$\langle \tilde{T}_i(t) \tilde{T}_j(t') \rangle_R \propto \delta(t - t')$$

Note $T_1 = T_2^+$

we take for a thermal state

$$\langle T_1(t) T_2(t') \rangle_R = \langle \tilde{T}(t) T^+(t') \rangle_R = \sum_{k,k'} g_k g_{k'} e^{-i\omega_k t} e^{+i\omega_{k'} t'} \underbrace{Tr_B(\rho_B(0) b_k b_{k'}^+)}_{\triangleq \bar{n}(\omega_k)+1 \text{ for } k=k'}$$

$$= \sum_k |g_k|^2 e^{-i(\omega_k)(t-t')} (\bar{n}(\omega_k, T) + 1)$$

assume that g_k is freq. independent.

$$\cong 2\pi\, g(\omega_s)^2 (\bar{n}+1) \cdot D(\omega)\ \delta(t-t')$$

$\gamma \equiv 2\pi |g(\omega_s)|^2 D(\omega_s)$

$$= \gamma \cdot (\bar{n}+1)\, \delta(t-t')$$

(γ: damping rate)

Thus we arrive at δ-correlated noise.

When carrying out the summation over all ij one obtain for $\dot{\rho}$ 16 Terms.

$$\dot{\rho} = \int_0^t dt' \{ [\hat{a}\hat{a}\rho(t) - \hat{a}\rho(t)\hat{a}] e^{-i\omega_0(t+t')} \langle T^+(t) T^+(t') \rangle_R + h.c.$$

$$+ \{ (\hat{a}^+\hat{a}^+\rho(t) - \hat{a}^+\rho(t)\hat{a}^+) e^{+i\omega_0(t+t')} \langle T(t) T(t') \rangle_R + h.c.$$

$$+ \{ (\hat{a}^+\hat{a}^+ \rho(t) - \hat{a}^+\rho(t)\hat{a}] e^{-i\omega(t-t')} \langle T^+(t) T(t') \rangle_R + h.c.$$

$$+ \{ \hat{a}^+\hat{a}\rho(t) - \hat{a}\rho(t)\hat{a}^+] e^{-i\omega_0(t-t')} \langle T(t) T^+(t') \rangle_R + h.c.$$

This yields now the Master Equation in rotating frame:

$$\dot{\tilde{\rho}} = \frac{\gamma}{2} (2\hat{a}\tilde{\rho}\hat{a}^+ - \hat{a}^+\hat{a}\rho - \rho\hat{a}^+\hat{a})$$

$$+ \gamma\bar{n} (\hat{a}\tilde{\rho}\hat{a}^+ - \hat{a}^+\rho\hat{a} - \hat{a}^+\hat{a}\rho - \rho\hat{a}\hat{a}^+)$$

Transforming back to Schrödinger picture

$$\dot{\tilde{\varrho}} = -i\omega[\hat{a}^{\dagger}\hat{a}, \varrho] + \frac{\gamma}{2}(\bar{n}+1)\{2\hat{a}\varrho\hat{a}^{+} - \hat{a}^{+}\hat{a}\varrho - \varrho\hat{a}^{+}\hat{a}\} + \frac{\gamma}{2}\bar{n}\{2\hat{a}^{+}\varrho\hat{a} - \hat{a}\hat{a}^{+}\varrho - \varrho\hat{a}\hat{a}^{+}\}$$

QUANTUM MASTER EQUATION (LINDBLAD FORM)

LINDBLAD FORM OF THE MASTER EQUATION, FOR THE DAMPED HARMONIC OSCILLATOR, OR (EQUIVALENT)

$$\dot{\varrho} = -i\omega[a^{+}a, \varrho] + \frac{\gamma}{2}(2a\varrho a^{+} - a^{+}a\varrho - \varrho a^{+}a) + \frac{\gamma}{2}\bar{n}(a\varrho a^{+} + a^{+}\varrho a - \varrho a a^{+})$$

QUANTUM MASTER EQUATION

More general expression for the QUANTUM MASTER EQUATION.

Coupling to the bath takes a form $\hat{H}_{int} = \sum g_k(\hat{b}_k^{+}\hat{C}) + (\hat{C}^{+}\hat{b}_k)g_k^{*}$

$\hat{C}$: jump operator (i.e a system operator(s) or a combination thereoff)

$$* \quad \dot{\tilde{\varrho}} = -\frac{i}{\hbar}[H_{sys}, \varrho] + \frac{\gamma}{2}(\bar{n}+1)\{2\hat{C}\varrho\hat{C}^{+} - \hat{C}^{+}\hat{C}\varrho - \varrho\hat{C}^{+}\hat{C}\} + \frac{\gamma}{2}\bar{n}\{2\hat{C}^{+}\varrho\hat{C} - \hat{C}\hat{C}^{+}\varrho - \varrho\hat{C}\hat{C}^{+}\}$$

Next: method of solution: Expectation values of Operators

Expectation values: $\langle a\rangle = Tr\{\hat{a}\varrho\} = Tr\{\varrho\hat{a}\}$

$$\langle\dot{a}\rangle = i\omega Tr(a^{+}aa\varrho - a\varrho a^{+}a) + \frac{\gamma}{2}Tr(2a^{2}\varrho a^{+} - aa^{+}a\varrho - a\varrho a^{+}a) + \frac{\gamma}{2}\bar{n}\,Tr\{a^{2}\varrho a^{+} + aa^{+}\varrho a - \varrho aa^{+}\}$$

$$= i\omega\, Tr\{\underbrace{(aa^{+} - a^{+}a)}_{=+1}a\varrho\} + \frac{\gamma}{2}Tr[2a^{+}aa\varrho \underbrace{- aa^{+}a\varrho - aa^{+}\varrho}_{[a^{+}a - aa^{+}]a\varrho}] \quad \text{commutator } [\hat{a},\hat{a}^{+}] = 1$$

$$+ \frac{\gamma}{2}\bar{n}\,Tr[(a^{+}a - aa^{+})a\varrho + a(aa^{+} - a^{+}a)\varrho]$$

$$\boxed{\langle n(t)\rangle = \langle n\rangle(0)\, e^{-\gamma t} + \bar{n}(1 - e^{-\gamma t})}$$

$$\boxed{\langle\dot{a}\rangle = -\left(\frac{\gamma}{2} + i\omega_0\right)\langle a\rangle}$$

i.e $\langle a\rangle = \langle a(0)\rangle\, e^{(-\gamma/2 + i\omega_0)t}$

Master equation for the damped harmonic oscillator

Thermal equilibrium of the HO:

$$\boxed{\dot{\rho} = -i\omega_0[a^\dagger a,\rho] + \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \gamma\bar{n}(a\rho a^\dagger + a^\dagger\rho a - a^\dagger a\rho - \rho a a^\dagger)}$$

or Lindblad form of the master equation

$$\boxed{\dot{\rho} = i\omega_0[a^\dagger a,\rho] + \frac{\gamma}{2}(\bar{n}+1)\{2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\} \;\; \frac{\gamma}{2}\bar{n}\{\underbrace{2a^\dagger\rho a}_{III} - \underbrace{aa^\dagger\rho - \rho a a^\dagger}_{IV}\}}$$

Methods of Solution (−): Fock state Basis

The physical interpretation arises from the rate equations satisfied by the probabilities $p_n = \langle n|\rho|n\rangle$ for the oscillator to be found in state $|n\rangle$

$$\dot{p}_n = \gamma(\bar{n}+1)(n+1)p_{n+1} - \gamma\bar{n}np_n + \underbrace{\gamma\bar{n}np_{n-1}}_{\hat{=}\,III} - \underbrace{\gamma\bar{n}(n+1)p_n}_{\hat{=}\,IV\text{ term}}$$

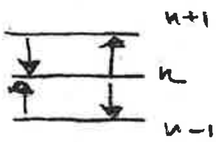
i.e. a birth death equation as before

since: $\langle n|a\rho a^\dagger|n\rangle = \sqrt{n+1}\,\langle n+1|\rho|n+1\rangle\sqrt{n+1}$

Thus

gain:	$g = \gamma/2(\bar{n}+1)\,n$ $\;t^-$
loss:	$r = \gamma/2\,\bar{n}\,(n+1)$ $\;t^+$

see: classical master equation earlier in class



We can solve the equation for the populations by the previously introduces BURGESS solution.

Note that $\rho_{nm} = P_{nm} = \langle n|\rho|m\rangle$ (i.e. the off diagonal elements) obey.

EQUATION FOR COHERENCES (off diagonal terms of ρ)

$$\boxed{\dot{\rho}_{nm} = \dot{P}_{nm} = \langle n|\dot{\rho}|m\rangle = i\omega\cdot m\,p_{nm} + \frac{\gamma}{2}(\bar{n}+1)\{2\sqrt{m+1}\sqrt{n+1}\,\rho_{n+1,m+1} - (n+m)\rho_{nm}] + \frac{\gamma}{2}\bar{n}\{2\sqrt{m-1}\sqrt{n-1}\,\rho_{n-1,m-1} - (n+m)\rho_{nm}\}}$$

off diagonal terms

In steady state the coherences decay $\dot{\rho}_{nm} \propto -(n+m)\rho_{nm}$

Thus $\rho^{ss}_{nm} = P^{ss}_{nm} = 0$

The stationary solution is (cf BURGESS solution before)

$$P_S(n) = \left(\frac{t^+}{t^-}\right)^n N$$

N: Normalization (geometric series)

$$\rightarrow \quad P_S(n) = \left(\frac{\bar{n}}{\bar{n}+1}\right)^n \frac{1}{1+\bar{N}}$$

Non Eq Stat. Phys

Master equation for the damped harmonic oscillator

Application: thermal equilibrium.

This distribution is a Boltzmann distribution, defining

$$\frac{\bar{n}}{1+\bar{n}} = \exp\left(\frac{-\hbar\omega}{k_B T}\right)$$

Thus: $\bar{n} = \dfrac{1}{\exp(\hbar\omega/k_B T)-1}$ (this relates $\bar{n}$ to Temperature T)

Thus the density matrix ς_{nm} for steady state is given by

$$p_{nn} = \varsigma_{nn} = p_s(n) = \frac{1}{\exp(\hbar\omega/k_B T)-1}\cdot \exp\left(\frac{-\hbar\omega}{k_B T}\right)^n$$

$$= \frac{1}{\exp(\hbar\omega/k_B T)-1}\exp\left(-n\frac{\hbar\omega}{k_B T}\right)$$

Thus equivalently:

$$\boxed{\varsigma^s = \frac{1}{\exp(\hbar\omega/k_B T)-1}\exp\left(-a^\dagger a\frac{\hbar\omega}{k_B T}\right)}$$

STEADY STATE DENSITY MATRIX OF A THERMAL STATE.

since $\langle n|\varsigma|n\rangle = \dfrac{1}{\exp(\frac{\hbar\omega}{k_B T})-1}\langle n|\underbrace{\exp(a^\dagger a\frac{\hbar\omega}{k_B T})}_{\hat{=}\ \exp(n\frac{\hbar\omega}{k_B T})}|n\rangle$ via Baker Hausdorff.

P- representation

(Glauber- Sudarshan P- representation)

[R. Glauber: Nobel Prize Physics: 2005

Chapter 6.2. "Quantum Noise"

Review.

We can use the coherent states ($a|\alpha\rangle = \alpha\cdot|\alpha\rangle$) as an over-complete basis (since: $|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha-\beta|^2)$) to express the density matrix ρ in the $|\alpha\rangle$ basis.

Note the completeness formula

$$1 = \frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha|$$

$$|\alpha\rangle = e^{-|\alpha|^2\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

COHERENT STATE

Outline of the proof:

assume arbitrary state vector $|\psi\rangle = \sum_n A_n |n\rangle$

Thus $\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha|A\rangle = \sum_n \int A_n |\alpha\rangle\langle\alpha|n\rangle d^2\alpha$

note that $\alpha \in \mathbb{C}$ thus $\alpha = \alpha_x + i\alpha_y$ and $\partial^2\alpha = d\alpha_x\, d\alpha_y$

Using $\langle\alpha|n\rangle$ and integration in polar coordinates

$\alpha = re^{i\theta}$ $\quad \partial^2\alpha = r\,dr\,d\theta$ yields

$$\ldots = \frac{1}{\pi}\sum_{m,n} \int A_n e^{-r^2} (r^{n+m}) e^{-i(m-n)} (n!m!)^{-1/2} |m\rangle r\,dr\,d\theta$$

$$= [\ldots] = 1$$

P- representation.

$$\rho = \int d^2\alpha\, P(\alpha,\alpha^*)\, |\alpha\rangle\langle\alpha|$$

QUASI-PROBABILITY DISTRIBUTION

$\alpha = x+iy$

- The values can become negative

This representation can be used to convert the Q-O-Master Eq to a FOKKER PLANCK EQUATION.

using :

$$\dot{\rho} = -\frac{\kappa}{2}\bar{n}_{th}(a a^\dagger\rho - 2a^\dagger\rho a + \rho a a^\dagger) - \frac{\kappa}{2}(\bar{n}_{th}+1)(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a)$$

leads to

$$\int \dot{P}(\alpha,\alpha^*,t)|\alpha\rangle\langle\alpha|\, d^2\alpha = -\frac{\kappa\bar{n}_{th}}{2}\int P(\alpha,\alpha^*,t)\left[a a^\dagger|\alpha\rangle\langle\alpha| - 2a^\dagger|\alpha\rangle\langle\alpha|a + |\alpha\rangle\langle\alpha|a a^\dagger\right] - \frac{\kappa}{2}(\bar{n}_{th}+1)\int P(\alpha,\alpha^*,t)\left[a^\dagger a|\alpha\rangle\langle\alpha| - 2a|\alpha\rangle\langle\alpha|a^\dagger + |\alpha\rangle\langle\alpha|a^\dagger a\right]d^2\alpha$$

It follows that

- $a|\alpha\rangle\langle\alpha| = \alpha|\alpha\rangle\langle\alpha|$

- $a^\dagger|\alpha\rangle\langle\alpha|$:

$$a^\dagger \rho = \int d^2\alpha \; a^\dagger P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha| = [\ldots]$$ (expand in term of coherent states)

$$= \int \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha,\alpha^*)\, d^2\alpha \qquad \text{i.e} \quad a^\dagger\rho \rightarrow \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha,\alpha^*)$$

(as in previous homeworks).

- $\rho a = |\alpha\rangle\langle\alpha|a \Leftrightarrow \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha,\alpha^*)$

- $\rho a^\dagger \Leftrightarrow \alpha^* P(\alpha,\alpha^*)$

Thus this yields after lengthy but simple calculation.

$$\boxed{\frac{d}{dt}\hat{P}(\alpha,\alpha^*,t) = \frac{\kappa}{2}\left[\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right] P(\alpha,\alpha^*,t) + \kappa\, \bar{n}_{TH}\left[\frac{\partial^2 P(\alpha,\alpha^*,t)}{\partial\alpha\,\partial\alpha^*}\right]}$$

Assume that the oscillator is initally in a coherent state

$$P(\alpha,\alpha^*,0) = \delta(\alpha - \alpha_0) \qquad \text{note } \alpha \in \mathbb{C}$$

We solve the F.P equation with form

$$P(\alpha,\alpha^*,t) = \exp(-a(t) + b(t)\alpha + c(t)\alpha^* - d(t)\,\alpha\alpha^*)$$

This yields after lengthy calculation (e.g Scully Quantum Optics Chapter 8)

$$\boxed{\begin{aligned} P(\alpha,\alpha^*,t) &= \frac{1}{\pi D(t)} \exp\left(\frac{|\alpha - \alpha_0 u(t)|^2}{D(t)}\right) \\ D(t) &= \bar{n}_{TH}(1 - e^{-\kappa t}) \\ \alpha_0 u(t) &= \alpha_0\, e^{-\kappa t/2 - i\omega t} \end{aligned}}$$

First calculated by Glauber* Phys. Rev.

(* Nobel Prize Physics 2005)

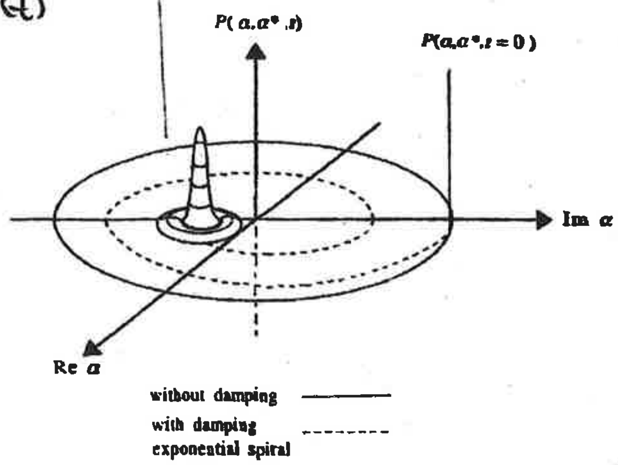


Fig. 8.1 The *P*-representatio for the complex amplitude of a harmonic oscillator mode damped by a thermal bath. The harmonic oscillator mode starts at *t* = 0 in a pure coherent sate |α⟩ and the mean value of the amplitude moves on an exponential spiral decreasing steadily in modulus, while its dispersion increases.